The Coupling Surface Method for the Solution of Magneto-Quasi-Static Problems

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In this paper we present a novel methodology for decoupling the solution of the Magneto-Quasi-Static (MQS) equations in subdomains, through a coupling surface and related equivalent currents. In particular, the solution outside the coupling surface is "condensed" with a suitable boundary condition. Different formulations can be used in each subdomain, hence allowing the most convenient approach to be used in each case. A simple test case is presented, showing the effectiveness of the method.

***Index Terms*—Coupling surface, Magneto-Quasi-Static limit**

I. Introduction

THE MAGNETO-QUASI-STATIC (MQS) problem is the most difficult to be solved numerically, among the mathematical models approximating Maxwell equations in static and quasi-static limits (i.e. when one or both the time derivatives can be neglected). This is confirmed by the great number of different formulations developed [1] and implemented in research and commercial codes.

The complexity is greatly exacerbated whenever nonlinear (e.g. ferromagnetic) materials are present in the domain and/or multiphysics problems must be treated (e.g. when a fusion plasma interacts with conductors [2]). In such cases, however, it may happen that the "complex" subdomain (nonlinear or multiphysics) is spatially limited, while the rest of the domain has more "simple" properties (e.g. linear conducting materials surrounded by vacuum). If this is the case, one can think to use different approaches in the various subdomains, in particular trying to exploit the "simplicity" wherever possible.

The present paper presents a novel methodology giving an answer in this respect. Assuming that the exterior (unbounded) part of the solution domain includes only linear nonmagnetic conductors, an integral formulation in the stream of [3] is used, in order to "condense" the effect of this part of the domain on the rest - in some sense, the proposed approach, detailed in Section II, can be seen as an analogue to Thevenin's theorem in circuit theory. A simple test case is also presented in Section III, to show the effectiveness of the method. In the full paper, other examples will be presented which are relevant to applications and that will highlight the theoretical properties and soundness of the method.

II. Formulation

Let us assume that the MQS (eddy currents) equations must be solved in an unbounded domain Ω.We suppose that Ω can be partitioned in two subdomains: Ω_i and Ω_e. We suppose that Ω_e is unbounded, including only linear non-magnetic conducting materials. Conversely, Ω_i is bounded, but we make no assumptions on the kind of materials inside Ω_i. We call S (the coupling surface) the boundary of Ω_i, separating the two domains.

The key point is to find two equivalent surface current densities: $\mathbf{J}_{Se}$ is able to produce the same magnetic field <u>outside</u> S as all the sources inside S; $\mathbf{J}_{Si}$ is able to produce the same magnetic field <u>inside</u> S as all the sources outside S. Adopting the approach depicted in [3], we give a finite elements discretization of S and we expand such current densities in terms of the curl of edge elements $\mathbf{N}_k$:

$$\mathbf{J}_{Se} = \sum_k I_{Se}(k) \nabla \times \mathbf{N}_k, \quad \mathbf{J}_{Si} = \sum_k I_{Si}(k) \nabla \times \mathbf{N}_k \tag{1}$$

The number and the choice of basis functions, related to the edges of the mesh, must be carefully done in order to ensure a correct representation of the current density.

Given the sources inside S, the quantity $\mathbf{J}_{Se}$ can be found by supposing that the magnetic vector potential produced by $\mathbf{J}_{Se}$ (denoted as $\mathbf{A}_{eq}$) is equal to the one produced by the sources inside S (indicated as $\mathbf{A}_i$). Given the uniqueness properties of the MQS problem, with suitable gauge conditions, it is sufficient to impose (in weak form) this condition on S to guarantee that it also holds in all Ω_e:

$$\int_S \mathbf{A}_{eq} \cdot \mathbf{w}\, dS = \int_S \mathbf{A}_i \cdot \mathbf{w}\, dS \tag{2}$$

where $\mathbf{w}$ is a suitable weighting function. From the physical point of view, we are assuming that S is made of a perfectly conducting material, in which a current is induced, able to perfectly cancel (outside S) the magnetic field of the sources inside S. Adopting the Galerking approach and using the Biot-Savart integral to express the vector potential, (2) becomes:

$$\begin{gathered} \underline{\underline{L}}_{SS}\, \underline{I}_{Se} = \underline{U}_i \\ L_{SS}(h,j) = \frac{\mu_0}{4\pi} \int_S \int_S \frac{\nabla \times \mathbf{N}_h(\mathbf{r}) \cdot \nabla \times \mathbf{N}_j(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dS\, dS' \\ U_i(k) = \int_S \nabla \times \mathbf{N}_k \cdot \mathbf{A}_i dS \end{gathered} \tag{3}$$

With a similar reasoning, we can get:

$$\begin{gathered} \underline{\underline{L}}_{SS}\, \underline{I}_{Si} = \underline{U}_e \\ U_e(k) = \int_S \nabla \times \mathbf{N}_k \cdot \mathbf{A}_e dS \end{gathered} \tag{4}$$

where $\mathbf{A}_e$ is the magnetic vector potential of all the sources outside S. Given the assumptions reported above about Ω_e, it is convenient to use an integral formulation to find $\mathbf{A}_e$. In particular, we follow [3]: we give a finite elements discretization of all non-magnetic conductors V in Ω_e, we describe the current density in V in terms of discrete DoF $\underline{I}_V$ and we impose Ohm's law in weak form with the Galerkin approach, giving rise to:

$$\underline{\underline{L}}_{VV}\frac{d\underline{I}_V}{dt}+\underline{\underline{R}}_{VV}\,\underline{I}_V+\underline{\underline{L}}_{VS}\frac{d\underline{I}_{Se}}{dt}=\underline{b}$$
$$R_{i,j}=\int_{V_c}\nabla\times\mathbf{N}_i\cdot\eta\cdot\nabla\times\mathbf{N}_j dV \qquad (5)$$

where the inductance terms have similar definitions as in (3) and $\underline{b}$ is a forcing term.

Solving (5) with an implicit time stepping, we have:

$$\underline{I}_V=-\underline{\underline{Z}}_{VV}^{-1}\underline{\underline{L}}_{VS}\underline{I}_{Se}+\underline{c} \qquad (6)$$

where $\underline{\underline{Z}}_{VV}=\underline{\underline{L}}_{VV}+\Delta t\,\underline{\underline{R}}_{VV}$ and $\underline{c}$ is a known term. From (6):

$$\underline{U}_e=\underline{\underline{L}}_{SV}\underline{I}_V+\underline{a}_e=-\underline{\underline{L}}_{SV}\underline{\underline{Z}}_{VV}^{-1}\underline{\underline{L}}_{VS}\underline{I}_{Se}+\underline{d} \qquad (7)$$

where $\underline{a}_e$ and $\underline{d}$ are known terms. Combining (3), (4), (7):

$$\underline{I}_{Si}=-\underline{\underline{L}}_{SS}^{-1}\underline{\underline{L}}_{SV}\underline{\underline{Z}}_{VV}^{-1}\underline{\underline{L}}_{VS}\underline{\underline{L}}_{SS}^{-1}\underline{U}_i+\underline{e}=\underline{\underline{K}}\,\underline{U}_i+\underline{e} \qquad (8)$$

where $\underline{\underline{K}}$ is a symmetric matrix and $\underline{e}$ is a known term.

Equation (8) is an affine relation between $\underline{I}_{Si}$ and $\underline{U}_i$, which in a sense "condense" what is present in Ω_e, thanks to the assumptions made and to the properties of the integral formulation used. Hence, (8) can be seen as a suitable boundary condition that can be used to solve the problem only in Ω_i, but self-consistently taking into account also the rest of the domain. The solution of the problem in Ω_i with boundary condition (8) can be in principle carried out with any formulation, regardless of how we get to (8).

III. Example of application

Although the proposed method is most effective when the solution of the interior problem in Ω_i is "difficult" (e.g. including nonlinear materials, or with multiphysics problems like interaction with fusion plasmas [2]), here we present a simple example in which also in Ω_i only nonmagnetic conducting materials are present, like in Ω_e. This example hence can be also seen in the frame of the approach proposed in [4], which quantifies the error made when neglecting the effects of the equivalent current. In this case, the same formulation as above can be used also in Ω_i, giving rise to:

$$\underline{\underline{L}}_{DD}\frac{d\underline{I}_D}{dt}+\underline{\underline{R}}_{DD}\,\underline{I}_D+\underline{\underline{L}}_{DS}\frac{d\underline{I}_{Si}}{dt}=\underline{b}_D$$
$$\underline{U}_i=\underline{\underline{L}}_{SD}\underline{I}_D+\underline{a}_D \qquad (9)$$

where the suffix "D" stands for the domains inside S and $\underline{a}_D$ is related to impressed sources inside S. Combining (9) and (8):

$$\underline{\underline{L}}_{DD}^{*}\frac{d\underline{I}_D}{dt}+\underline{\underline{R}}_{DD}\,\underline{I}_V=\underline{b}_D^{*} \qquad (10)$$

with obvious definitions. In other words, we can solve a reduced problem, only on the interior domain, in which the presence of the rest of the domain is taken into account via a modified inductance matrix $\underline{\underline{L}}_{DD}^{*}=\underline{\underline{L}}_{DD}+\underline{\underline{L}}_{DS}\underline{\underline{K}}\,\underline{\underline{L}}_{SD}$ and a modified forcing term $\underline{b}_D^{*}$.

Figure 1 shows an example in which we consider two conducting blocks and one driving coil, fed with a time-varying voltage. The coupling surface (Fig. 2) is a cube enclosing the lower block (interior domain). The exterior (unbounded) domain comprises the higher block and the driving coil. Figure 1 also illustrates the current density induced in the two blocks at a sample time instant; Fig. 2 shows instead the equivalent current $\underline{I}_{Se}$ on the coupling surface. We computed the solution both with the standard approach (i.e. considering both the blocks together) and using the proposed coupling scheme (i.e. only in the lower block, with the modified inductance matrix). The maximum difference between the two solutions is around 5%. This error can be improved by increasing the discretization level of the coupling surface. This work was supported in part by Italian MIUR under PRIN grant 2010SPS9B3.

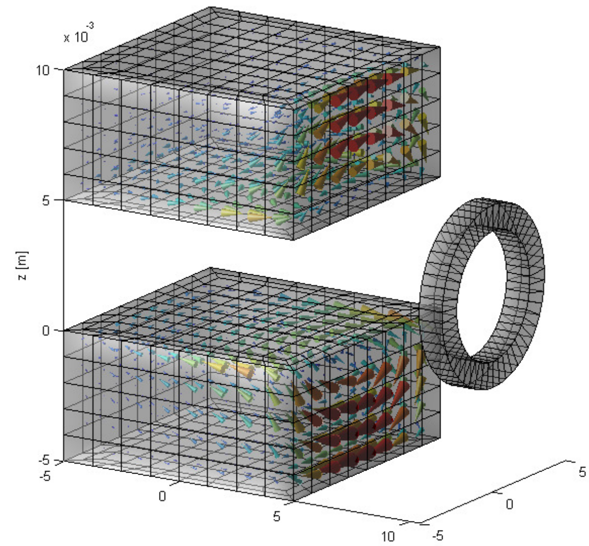

Fig. 1. Current density pattern induced in the two conducting block.

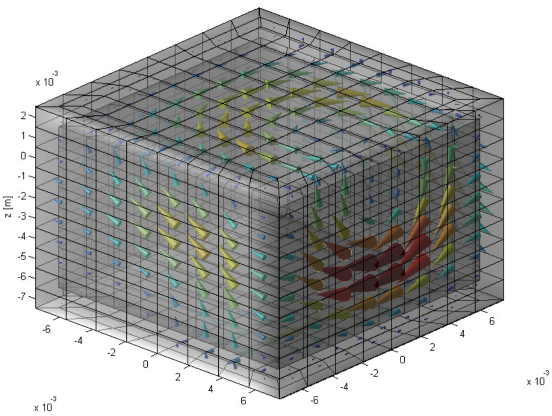

Fig. 2. Equivalent current on the coupling surface

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